Chapter 7: Delta-Hedging and Risk-Neutral Pricing

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Preview

This chapter introduces two pricing approaches of contingent claims based on an underlying asset. We first consider an analytic approach using the delta-hedging method. By constructing a replicating portfolio, we see that the price of the contingent claim satisfies a PDE, known as the Black-Scholes equation. In the second approach, we consider a probabilistic approach using the risk-neutral pricing method. Using Girsanov's theorem, we can compute the price of the contingent claim by its expected discounted value under the risk-neutral measure.

Key topics in this chapter:

1. Delta-hedging and Black-Scholes equation;
2. Risk-neutral measure and Girsanov's theorem;
3. Risk-neutral pricing.

1 Delta-Hedging and Black-Scholes Equations

Delta-hedging and portfolio replication is a pricing approach of contingent claims by solving an associated partial differential equation (PDE), which is known as the Black-Scholes equation. In this section, we fix a time $T > 0$ and let $\{S_t\}_{t\in[0,T]}$ be a stock price process, which follows a geometric Brownian motion:

$$dS_t = \mu\, dt + \sigma\, dB_t,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the rate of return and the volatility of the risky asset, respectively.

A ***contingent claim*** is a derivative whose future payoff depends on the underlying risky asset $\{S_t\}_{t\in[0,T]}$. Let V_T be an $\mathcal{F}_T$-measurable random variable that represents the payoff of a derivative security at time T, based on the stock price process $\{S_t\}_{t\in[0,T]}$. The payoff V_T may be path-dependent, meaning it can depend on the entire trajectory of $\{S_t\}_{t=0}^{T}$, rather than solely on the terminal value S_T. Examples of derivatives and their payoffs include:

1. **European options**:
    (a) Call: $(S_T - K)^+$
    (b) Put: $(K - S_T)^+$
2. **Asian options**:
    (a) Call: $\left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+$
    (b) Put: $\left(K - \frac{1}{T}\int_0^T S_t\,dt\right)^+$

We exclude American options, since they allow investors to choose an optimal exercise time, leading to an optimal stopping problem that is beyond the scope of this course. Although Asian options cannot be priced using the Black-Scholes equation, it can be priced using risk-neutral method later in this chapter.

The core idea of delta-hedging/replicating portfolio is to construct a portfolio strategy $\{\Delta_t\}_{t\in[0,T]}$ such that the portfolio value matches the claim's value throughout the interval $t \in [0,T]$, which yields a partial differential equation (PDE) that governs the evolution of the claim's price. In this section, we assume that $V_T = f(S_T)$, i.e., the payoff depends on the final price of the risky asset. We assume that there exists a function $V : [0,T]\times\mathbb{R} \to \mathbb{R}$, such that $V(t,s)$ represents that price of the contingent claim at time t given that $S_t = s$.

1.1 Black-Scholes Equation

We construct a self-financing replicating portfolio $\{X_t\}_{t\in[0,T]}$ by:

(i) Choosing an initial capital X_0 at $t = 0$;

(ii) At each time $t \in [0,T]$, holding Δ_t units of the stock, with the remaining amount $X_t - \Delta_t S_t$ invested at or borrowed from the risk-free rate r.

We now derive the dynamics of the replicating portfolio $\{X_t\}_{t\in[0,T]}$. A portfolio is called ***self-financing*** if its value changes only due to gains and losses from the assets held, without any additional infusion or withdrawal of capital. Mathematically, we can write the portfolio value as

$$X_t = \Delta_t S_t + \beta_t,$$

where $\beta_t = X_t - \Delta_t S_t$ is the bank account position. Differentiating gives

$$dX_t = \Delta_t\,dS_t + S_t\,d\Delta_t + d\beta_t.$$

The ***self-financing condition*** requires that any change in Δ_t is exactly offset by the bank account, i.e.

$$d\beta_t = r\beta_t\,dt - S_t\,d\Delta_t = r(X_t - \Delta_t S_t)\,dt - S_t\,d\Delta_t.$$

Substituting, the $S_t\, d\Delta_t$ terms cancel, leaving

$$
\begin{aligned}
dX_t &= \underbrace{\Delta_t\, dS_t}_{\text{proceeds from holding the stock}} + \underbrace{r(X_t - \Delta_t S_t)\, dt}_{\text{proceeds from lending/borrowing at } r} \\
&= \Delta_t \left(S_t \mu\, dt + S_t \sigma\, dB_t\right) + r(X_t - \Delta_t S_t)\, dt \\
&= (rX_t + \Delta_t(\mu - r)S_t)\, dt + \Delta_t \sigma S_t\, dB_t. \qquad (1)
\end{aligned}
$$

By Itô's lemma, the discounted portfolio value $\{e^{-rt}X_t\}_{t\in[0,T]}$ satisfies the following SDE:

$$
\begin{aligned}
d(e^{-rt}X_t) &= -re^{-rt}X_t\, dt + e^{-rt}\, dX_t \\
&= e^{-rt}\Delta_t S_t \left((\mu - r)\, dt + \sigma\, dB_t\right). \qquad (2)
\end{aligned}
$$

Next, we consider the evolution of the price of the contingent claim. At time t, this price is given by $V(t, S_t)$. Applying Itô's lemma to the process $\{V(t, S_t)\}_{t\in[0,T]}$, we have

$$
\begin{aligned}
dV(t, S_t) &= V_t(t, S_t)\, dt + V_S(t, S_t)\, dS_t + \frac{1}{2}V_{SS}(t, S_t)\, d\langle S\rangle_t \\
&= \left[V_t(t, S_t) + V_S(t, S_t)S_t\mu + \frac{1}{2}\sigma^2 S_t^2 V_{SS}(t, S_t)\right] dt + V_S(t, S_t)\sigma S_t\, dB_t.
\end{aligned}
$$

Therefore, the discounted price of the contingent claim, $\{e^{-rt}V(t, S_t)\}_{t\in[0,T]}$, is given by

$$
\begin{aligned}
d(e^{-rt}V(t, S_t)) &= -re^{-rt}V(t, S_t)\, dt + e^{-rt}dV(t, S_t) \\
&= e^{-rt}\left[-rV(t, S_t) + V_t(t, S_t) + V_S(t, S_t)S_t\mu + \frac{1}{2}\sigma^2 S_t^2 V_{SS}(t, S_t)\right] dt \\
&\quad + e^{-rt}V_S(t, S_t)\sigma S_t\, dB_t. \qquad (3)
\end{aligned}
$$

The portfolio $\{X_t\}_{t\in[0,T]}$ under the strategy $\{\Delta_t\}_{t\in[0,T]}$ is said to be a ***replicating portfolio*** of the contingent claim if, for any $t \in [0, T]$, $X_t = V(t, S_t)$. To construct a replicating portfolio, we will need $V(0, s) = X_0$, and for any $t \in [0, T]$, $dX_t = dV(t, S_t)$, or equivalently,

$$
d(e^{-rt}X_t) = d(e^{-rt}V(t, S_t)). \qquad (4)
$$

Using (4) and equating the diffusion term of (2) and (3), we have

$$
\Delta_t \sigma S_t = V_S(t, S_t)\sigma S_t \Rightarrow \boxed{\Delta_t = V_S(t, S_t)}. \qquad (5)
$$

Further using (5), and equating the drift term of (2) and (3), we obtain

$$
\begin{aligned}
\Delta_t S_t(\mu - r) &= -rV(t, S_t) + V_t(t, S_t) + \mu S_t V_S(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 V_{SS}(t, S_t) \\
V_S(t, S_t)(\mu - r) &= -rV(t, S_t) + V_t(t, S_t) + \mu S_t V_S(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 V_{SS}(t, S_t)
\end{aligned}
$$

$$0 = V_t(t, S_t) + rS_tV_S(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 V_{SS}(t, S_t) - rV(t, S_t).$$

This leads to the ***Black-Scholes equation***:

Theorem 1.1 Under the Black-Scholes model, the price $V(t, s)$ of the contingent claim $V_T = f(S_T)$ at time $t \in [0, T]$ when $S_t = s$ is the solution of the ***Black-Scholes partial differential equation***:

$$\begin{cases} V_t(t, s) + rsV_S(t, s) + \frac{1}{2}\sigma^2 s^2 V_{SS}(t, s) - rV(t, s) = 0, \\ V(T, s) = f(s). \end{cases} \tag{6}$$

The replicating strategy is given by $\Delta_t = V_S(t, S_t)$.

The Black-Scholes equation is a *parabolic PDE*. To completely solve the equation, one usually needs two boundary conditions at $s = 0$ and $s = \infty$ in addition to the terminal condition $V(T, s) = f(s)$.

1. Boundary condition at $s = 0$: substituting $s = 0$ into (1.2), we obtain $V_t(t, 0) = rV(t, 0)$. Solving this ODE yields

   $$V(t, 0) = e^{rt}V(0, 0).$$

   In particular, $f(0) = V(T, 0) = e^{rT}V(0, 0)$, so that $V(0, 0) = e^{-rT}f(0)$. Therefore,

   $$V(t, 0) = e^{-r(T-t)}f(0), \ t \in [0, T].$$

   For call options, $f(0) = (0 - K)_+ = 0$, so that $V(t, 0) = 0$.

2. Boundary condition as $s \to +\infty$: the boundary condition as $s \to +\infty$ is customized based on the payoff of the contingent claim. For call options $f(s) = (s - K)_+$, we generally set

   $$\lim_{s \to +\infty} \left(V(t, s) - \left(s - e^{-r(T-t)}K\right)\right) = 0, \ t \in [0, T].$$

   This reflects that when the option is deep in the money, its value behaves like the difference between the forward price of the stock and the discounted strike, i.e., $V(t, s) \approx s - e^{-r(T-t)}K$.

1.2 Price of European Options

We now derive the expression $V(t, s)$ for an European call option, i.e., $V(T, s) = (s - K)_+$, where $K > 0$ is the strike price. Under this setting, the Black-Scholes equation reads

$$\begin{cases} V_t(t, s) + rsV_S(t, s) + \frac{1}{2}\sigma^2 s^2 V_{SS}(t, s) - rV(t, s) = 0, \\ V(T, s) = (s - K)_+. \end{cases}$$

To solve the PDE, we consider the following transformation:

$$\tau := T - t, \; x = \ln\left(\frac{s}{K}\right), \; w(\tau, x) := V(t, s) = V\left(T - \tau, Ke^x\right).$$

Using the change of variables, we compute the derivatives of V in terms of w. Since $V(t,s) = w(\tau, x)$ with $\tau = T - t$ and $x = \ln(s/K)$, we have $\frac{d\tau}{dt} = -1$, $\frac{dx}{ds} = \frac{1}{s}$, and thus

$$\begin{aligned}
V_t(t,s) &= -w_\tau(\tau, x), \; V_s(t,s) = \frac{1}{s} w_x(\tau, x), \\
V_{ss}(t,s) &= -\frac{1}{s^2} w_x(\tau, x) + \frac{1}{s^2} w_{xx}(\tau, x) = \frac{1}{s^2}\left(w_{xx}(\tau, x) - w_x(\tau, x)\right).
\end{aligned}$$

Substitute these into the Black–Scholes PDE, we have

$$\begin{aligned}
0 &= V_t + rsV_s + \frac{1}{2}\sigma^2 s^2 V_{ss} - rV \\
&= -w_\tau + rs\left(\frac{1}{s} w_x\right) + \frac{1}{2}\sigma^2 s^2 \left[\frac{1}{s^2}\left(w_{xx}(\tau, x) - w_x(\tau, x)\right)\right] - rw \\
&= -w_\tau + \left(r - \frac{1}{2}\sigma^2\right) w_x + \frac{1}{2}\sigma^2 w_{xx} - rw \\
\Rightarrow w_\tau &= \left(r - \frac{1}{2}\sigma^2\right) w_x + \frac{1}{2}\sigma^2 w_{xx} - rw,
\end{aligned}$$

with the initial condition $w(0, x) = (Ke^x - K)_+ = K(e^x - 1)_+$.

We further consider a change of variable to transform the PDE for w to a standard heat equation. To this end, we consider the *ansatz*

$$w(\tau, x) = e^{Ax + B\tau}\, u(\tau, x),$$

with constants A, B to be chosen. Compute the derivatives

$$\begin{aligned}
w_\tau &= e^{Ax+B\tau}\big(u_\tau + Bu\big), \\
w_x &= e^{Ax+B\tau}\big(u_x + Au\big), \\
w_{xx} &= e^{Ax+B\tau}\big(u_{xx} + 2Au_x + A^2 u\big).
\end{aligned}$$

Substituting these into the PDE of w and canceling the common factor $e^{Ax+B\tau}$ gives

$$\begin{aligned}
u_\tau + Bu &= \frac{1}{2}\sigma^2\big(u_{xx} + 2Au_x + A^2 u\big) + \Big(r - \frac{1}{2}\sigma^2\Big)\big(u_x + Au\big) - ru \\
\Rightarrow u_\tau &= \frac{1}{2}\sigma^2 u_{xx} + \Big(\sigma^2 A + r - \frac{1}{2}\sigma^2\Big) u_x + \left(\frac{1}{2}\sigma^2 A^2 + \left(r - \frac{1}{2}\sigma^2\right) A - r - B\right) u.
\end{aligned}$$

By choosing

$$A = \frac{1}{2} - \frac{r}{\sigma^2},\ B = \frac{1}{2}\sigma^2 A^2 + \left(r - \frac{1}{2}\sigma^2\right) A - r,$$

we arrive at the following heat equation.

$$\begin{cases} u_\tau = \frac{1}{2}\sigma^2 u_{xx}, \\ u(0,x) = e^{-Ax} w(0,x) = Ke^{-Ax}(e^x - 1)_+. \end{cases}$$

The general solution of the heat equation is given by convolution with the Gaussian kernel:

$$\begin{aligned} u(\tau, x) &= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{2\sigma^2\tau}\right) u(0,y)\, dy \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{2\sigma^2\tau}\right) e^{-Ay}(e^y - 1)_+\, dy. \end{aligned}$$

Upon solving the equation, we obtain the formula:

$$\boxed{V(t,s) = sN(d_1(T-t,s)) - Ke^{-r(T-t)}N(d_2(T-t,s)),}$$

where

$$d_1(\tau, x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},\ d_2(\tau, x) = d_1 - \sigma\sqrt{\tau},$$

and $N(\cdot)$ denotes the cumulative distribution function of the standard normal variable.

Using the pricing formula for European call options, we can compute the price of the European put option using the ***put-call parity***.

> **Theorem 1.2** Let C and P denote the price of an European call option and European put option with expiration time T, strike price K, on an underlying asset with price S respectively. Then,
>
> $$C + Ke^{-rT} = S + P.$$

Proof. Construct a portfolio by purchasing a unit of call option and lending Ke^{-rT} at the riskfree rate r. The price of the portfolio is $C + Ke^{-rT}$. At time T, the payoff of the portfolio is $(S_T - K)_+ + Ke^{-rT} \cdot e^{rT} = \max\{S_T, K\}$.

Now, construct another portfolio with with a long position of a unit of put option and a long position of a unit of the underlying asset. The price of the portfolio is $S + P$. At time T, the payoff of the portfolio is $(K - S_T)_+ + S_T = \max\{S_T, K\}$.

Since the two portfolios share the same payoff, by the law of one price, the costs of the portfolios must be the same, i.e., $C + Ke^{-rT} = S + P$.

□

Using the put-call parity and the fact the $N(-x) = 1 - N(x)$, the put option price $V^P(t,s)$ at time t with $S_t = s$ is given by

$$
\begin{aligned}
V^p(t,s) &= V(t,s) + Ke^{-r(T-t)} - s \\
&= s\left(N(d_1(T-t,s)) - 1\right) - Ke^{-r(T-t)}\left(N(d_2(T-t,s)) - 1\right) \\
&= \boxed{Ke^{-r(T-t)}N(-d_2(T-t,s)) - sN(-d_1(T-t,s)).}
\end{aligned}
$$

2 Risk Neutral Measure: Motivations from Binomial Tree Model

The risk-neutral pricing method is an alternative approach to delta-hedging. The core idea is to compute the price of the contingent claim using its expected discount value under an appropriate probability measure. In this section, we recall the risk-neutral pricing of a contingent claim based on the price of a risky asset under a binomial tree model.

Let $S_0 > 0$ be the current stock price. Over one period Δt, the stock moves to $S^u = uS_0$ or $S^d = dS_0$ with $u > 1 > d > 0$. Let $B_0 = 1$ be the money market account and $B_1 = e^{r\Delta t}$ with $r \geq 0$ being the continuously compounded risk-free interest rate. A (one-period) contingent claim H pays $H^u := H(S^u)$ in the up state and $H_d := H(S_d)$ in the down state at time 1 (i.e., at $t = \Delta t$).

Suppose under the real-world (physical) measure $\mathbb{P}$, the up move occurs with probability $p \in (0,1)$. A tempting price formula is

$$V_0 \stackrel{?}{=} e^{-r\Delta t}\,\mathbb{E}^{\mathbb{P}}[H] = e^{-r\Delta t}\big(pH_u + (1-p)H_d\big).$$

However, this method incorrectly uses r as the discount factor under the real-world measure. Indeed, under $\mathbb{P}$, a stochastic discount factor that incorporates the risk premium would be required. Hence, arbitrage-free prices cannot be obtained this way. Instead, we use the method of replicating portfolio to price the contingent claim as follows.

Let Δ denote the number of units of the risky asset and B the amount invested in the bank account. The initial portfolio value is $\Pi_0 = \Delta S_0 + B$. We seek Δ and B such that the portfolio replicates the contingent claim, i.e., its value at time 1 coincides with the payoff in both states. This requires solving

$$\Pi_1^{(u)} = \Delta S^u + Be^{r\Delta t} = H^u, \quad \Pi_1^{(d)} = \Delta S^d + Be^{r\Delta t} = H^d$$

replicates the claim in both states. Solving the linear system,

$$
\begin{aligned}
\Delta &= \frac{H^u - H^d}{S^u - S^d} = \frac{H^u - H^d}{(u-d)S_0}, \\
B &= \frac{e^{-r\Delta t}(uH^d - dH^u)}{u-d}.
\end{aligned}
$$

Hence the arbitrage-free price is

$$V_0 = \Pi_0 = \Delta S_0 + B = e^{-r\Delta t}(qH_u + (1-q)H_d)\,,$$

where the ***risk-neutral probability*** q is defined by

$$q := \frac{e^{r\Delta t} - d}{u - d} \in (0,1),$$

provided that $d < e^{r\Delta t} < u$. Equivalently,

$$V_0 = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r\Delta t} H\right],$$

where $\widetilde{\mathbb{P}}$ is the (unique) risk-neutral measure with $\widetilde{\mathbb{P}}(\text{up}) = q$ and $\widetilde{\mathbb{P}}(\text{down}) = 1-q$. Therefore, it is only under the risk-neutral measure $\widetilde{\mathbb{P}}$ that the price of the contingent claim can be expressed as the expected discounted payoff, with r serving as the discount rate.

By construction of q,

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[S_{t_1}] = \mathbb{E}^{\widetilde{\mathbb{P}}}[S_{t_1} \mid \mathcal{F}_0] = qS_u + (1-q)S_d = \big(e^{r\Delta t} - d\big)\frac{S^u - S^d}{u-d} + S^d = e^{r\Delta t} S_0.$$

In general, for a multi-period model, the same argument yields, for $n = 0, 1, \ldots,$

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[S_{t_{n+1}} \mid \mathcal{F}_{t_n}\right] = e^{r\Delta t} S_{t_n}, \text{ so that } \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r(n+1)\Delta t} S_{t_{n+1}} \mid \mathcal{F}_{t_n}\right] = e^{-rn\Delta t} S_{t_n},$$

where $t_n = n\Delta t$. This implies that the *discounted* stock price $\{e^{-rn\Delta t} S_{t_n}\}$ is a martingale under $\widetilde{\mathbb{P}}$.

3 Risk-Neutral Measures

The binomial tree model in Section 2 suggests that in order to compute the price of a contingent claim as the expected discounted payoff at the risk-free rate, one must introduce an alternative measure $\widetilde{\mathbb{P}}$. Under this measure, the discounted stock price becomes a martingale. In this section, we fix a time $T > 0$ and construct a risk-neutral measure $\widetilde{\mathbb{P}}$ when the stock price process $\{S_t\}_{t\in[0,T]}$ follows a generalized geometric Brownian motion:

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dB_t,$$

where $\{\mu_t\}_{t\in[0,T]}$ and $\{\sigma_t\}_{t\in[0,T]}$ are adapted processes, and $\sigma_t > 0$ for all $t \in [0,T]$. We also let $\{D_t\}_{t\in[0,T]}$ be a stochastic discount factor given by

$$D_t = e^{-\int_0^t r_s\, ds},$$

where $\{r_t\}_{t\in[0,T]}$ is an adapted process representing the stochastic interest rate.

Consider the discounted stock price process, $\{D_t S_t\}_{t\in[0,T]}$, which by Itô's lemma (product rule; see Theorem 3.2 in Chapter 5), admits the following dynamics:

$$\begin{aligned} d(D_t S_t) &= S_t\,dD_t + D_t\,dS_t + d\langle D, S\rangle_t \\ &= -r_t S_t D_t\,dt + D_t\left(\mu_t S_t\,dt + \sigma_t S_t\,dB_t\right) \\ &= D_t S_t\left((\mu_t - r_t)dt + \sigma_t\,dB_t\right), \end{aligned} \tag{7}$$

Note that the cross-variation term vanishes since D_t has a zero quadratic variation. From (7), the discounted process $D_t S_t$ is not a martingale since its SDE is not driftless unless in the special case $\mu \equiv r$.

If we define a process $\{\widetilde{B}_t\}_{t\in[0,T]}$ by

$$\widetilde{B}_t := B_t + \int_0^t \theta_s\,ds, \text{ where } \theta_t := \frac{\mu_t - r_t}{\sigma_t}.$$

Then, we can rewrite the dynamics of $D_t S_t$ as

$$d(D_t S_t) = D_t S_t \sigma_t\left(\theta_t\,dt + dB_t\right) = D_t S_t \sigma_t\,d\widetilde{B}_t. \tag{8}$$

The process $\{\theta_t\}_{t\in[0,T]}$ is called the ***market price of risk***. Since $\widetilde{B}_t$ is not a Brownian motion under the real-world measure $\mathbb{P}$, this change of variable alone will not make the discounted process $D_t S_t$ a martingale under $\mathbb{P}$. However, if we introduce an alternative measure $\widetilde{\mathbb{P}}$ under which $\{\widetilde{B}_t\}_{t\in[0,T]}$ is a genuine Brownian motion, then $D_t S_t$ becomes a true martingale. This measure $\widetilde{\mathbb{P}}$ is called the ***risk-neutral measure***. To this end, we shall introduce ***Girsanov's theorem***.

3.1 Change of Measures

In Chapter 1, we briefly introduced the idea of absolute continuity of the distribution $\mathbb{P}_X$ of a random variable X with respect to the Lebesgue measure. In that case, $\mathbb{P}_X$ can be described using a probability density function. This concept naturally extends to the relationship between two general probability measures.

> **Definition 3.1** Let $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ be two probability measures on the measurable space $(\Omega, \mathcal{F})$. We say that $\widetilde{\mathbb{P}}$ is ***absolutely continuous*** with respect to $\mathbb{P}$, denoted by $\widetilde{\mathbb{P}} \ll \mathbb{P}$, if for any $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$, it holds that $\widetilde{\mathbb{P}}(A) = 0$. If $\widetilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \widetilde{\mathbb{P}}$, we say that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are ***equivalent***. In that case, we write $\mathbb{P} \sim \widetilde{\mathbb{P}}$.

From the previous example, we observed that the discounted stock price process $\{D_t S_t\}$ is not a martingale under the real-world measure. This motivates the search for an alternative measure under which it becomes a martingale, which is precisely the role of the risk-neutral measure. Changing the underlying probability measure alters the distribution of random variables, and to relate expectations under the two measures, we use the Radon–Nikodym derivative. This provides the key tool to express quantities such as $\mathbb{E}^{\widetilde{\mathbb{P}}}[X]$ in terms of the original measure $\mathbb{P}$.

Theorem 3.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a non-negative random variable such that $\mathbb{E}[Z] = 1$. Define $\widetilde{\mathbb{P}} : \mathcal{F} \to \mathbb{R}$ by

$$\widetilde{\mathbb{P}}(A) := \mathbb{E}[Z \mathbb{1}_A], \quad \text{for all } A \in \mathcal{F},$$

where the expectation is taken with respect to $\mathbb{P}$.
Then:

1. $\widetilde{\mathbb{P}}$ is a probability measure absolutely continuous with respect to $\mathbb{P}$, i.e., $\widetilde{\mathbb{P}} \ll \mathbb{P}$.
2. For any integrable random variable X such that $XZ \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, we have

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[XZ].$$

3. The function Z is called the ***Radon–Nikodym derivative*** of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, denoted

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

4. If in addition $Z > 0$ $\mathbb{P}$-a.s., then $\mathbb{P} \sim \widetilde{\mathbb{P}}$, and for any $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, we have

$$\mathbb{E}[Y] = \widetilde{\mathbb{E}}\left[\frac{Y}{Z}\right].$$

Proof.

1. We verify that $\widetilde{\mathbb{P}}$ is a probability measure:
   (i) $\widetilde{\mathbb{P}}(A) \in [0,1]$ for any $A \in \mathcal{F}$: $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{1}_A] \le \mathbb{E}[Z] = 1$, and $\widetilde{\mathbb{P}}(A) \ge 0$ since $Z \ge 0$ a.s.
   (ii) $\widetilde{\mathbb{P}}(\Omega) = 1$: $\widetilde{\mathbb{P}}(\Omega) = \mathbb{E}[Z\mathbb{1}_\Omega] = \mathbb{E}[Z] = 1$.
   (iii) Countable additivity: let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$. Then,

$$\widetilde{\mathbb{P}}\left(\cup_{n=1}^\infty A_n\right) = \mathbb{E}\left[Z \mathbb{1}_{\cup_{n=1}^\infty A_n}\right] = \mathbb{E}\left[Z \sum_{n=1}^\infty \mathbb{1}_{A_n}\right] = \sum_{n=1}^\infty \mathbb{E}[Z\mathbb{1}_{A_n}] = \sum_{n=1}^\infty \widetilde{\mathbb{P}}(A_n),$$

   where we have used the MCT in the second-to-last equality.

   To verify that $\widetilde{\mathbb{P}} \ll \mathbb{P}$, note that for any $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$, we have $Z\mathbb{1}_A = 0$ $\mathbb{P}$-a.s., which implies $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{1}_A] = 0$.

2. Suppose that $X = \mathbb{1}_A$, where $A \in \mathcal{F}$. Then,

$$\widetilde{\mathbb{E}}[X] = \int_\Omega \mathbb{1}_A(x) d\widetilde{\mathbb{P}}(x) = \widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{1}_A] = \mathbb{E}[XZ].$$

This verifies the formula for indicator random variables. By linearity, the formula also holds for any simple random variable. For general random variables, we can use the standard 2-step approach: prove the formula for any non-negative random variables using dyadic approximation and MCT, followed by any general random variables by considering the decomposition $X = X^+ - X^-$. We omit the details herein.

4. If $Z > 0$ $\mathbb{P}$-a.s, we have that $0 \le 1/Z < \infty$ $\mathbb{P}$-a.s., which is thus a well-defined random variable. In addition, $1/Z \in L^1(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, since

$$\widetilde{\mathbb{E}}\left[\frac{1}{Z}\right] = \mathbb{E}\left[Z \cdot \frac{1}{Z}\right] = 1.$$

Note that $\mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A] = \widetilde{\mathbb{E}}[\mathbb{1}_A/Z]$. Hence, using Statements 1 and 2 of the theorem, we have $\mathbb{P} \ll \widetilde{\mathbb{P}}$, $\mathbb{E}[Y] = \widetilde{\mathbb{E}}[Y/Z]$ for any $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, where the Radon–Nikodym derivative of $\mathbb{P}$ with respect to $\widetilde{\mathbb{P}}$ is given by

$$\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} = \frac{1}{Z}.$$

□

Example 3.1 Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$. Consider the random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ defined by $Z(\omega) = 2\omega$, $\omega \in [0,1]$.

(a) Show that $\widetilde{\mathbb{P}} : \mathcal{F} \to \mathbb{R}$ defined by $\widetilde{\mathbb{P}}(A) := \mathbb{E}[Z\mathbb{1}_A]$ is a probability measure.

(b) For any $a, b \in [0,1]$ with $b > a$, calculate $\widetilde{\mathbb{P}}([a,b])$.

Solution.

(a) Since $Z \ge 0$ $\mathbb{P}$-a.s., it suffices to show that $\mathbb{E}[Z] = 1$. Indeed,

$$\mathbb{E}[Z] = \int_0^1 2\omega \, d\omega = \omega^2\Big|_0^1 = 1. \tag{9}$$

Therefore, $\widetilde{\mathbb{P}}$ defines a probability measure.

(b) For any $0 \le a < b \le 1$,

$$\widetilde{\mathbb{P}}([a,b]) = \mathbb{E}[Z\mathbb{1}_{[a,b]}] = \int_a^b 2\omega \, d\omega = b^2 - a^2.$$

□

The following example illustrates how a drifted normal random variable becomes a standard normal variable under a change of measure, where the Radon–Nikodym derivative is given by an exponential function. This is closely related to Girsanov's theorem and the construction of a risk-neutral measure, which will be explored later in this course.

Example 3.2 Suppose that $X \sim \mathcal{N}(0,1)$ is a standard normal variable on the probability space $(\Omega, \mathcal{F}, \mathbb{F})$, i.e., X admits the following probability density function. Let $Y := X + \mu$, where $\mu \in \mathbb{R}$, and a non-negative random variable Z by

$$Z = \exp\left(-\mu X - \frac{\mu^2}{2}\right).$$

(a) Determine the distribution of Y on $(\Omega, \mathcal{F}, \mathbb{P})$.
(b) Show that $\widetilde{\mathbb{P}} : \mathcal{F} \to \mathbb{R}$ defined by $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{1}_A]$, $A \in \mathcal{F}$, is a probability measure.
(c) Determine the distribution of Y on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$.

<u>*Solution*</u>.
(a) Since $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(\mu, 1)$.
(b) Since $Z \geq 0$ $\mathbb{P}$-a.s., by Theorem 3.1, it suffices to verify that $\mathbb{E}[Z] = 1$. Indeed,

$$\mathbb{E}[Z] = e^{-\frac{\mu^2}{2}} \mathbb{E}[e^{-\mu X}] = e^{-\frac{\mu^2}{2}} e^{\frac{\mu^2}{2}} = 1,$$

where we have used the formula of the MGF for a standard normal variable.
(c) We compute the MGF of Y under $\widetilde{\mathbb{P}}$. For any $t \in \mathbb{R}$, using Theorem 3.1,

$$\widetilde{\mathbb{E}}[e^{tY}] = \mathbb{E}[e^{tY} Z] = \mathbb{E}\left[e^{t(X+\mu)} e^{-\mu X - \frac{\mu^2}{2}}\right] = e^{t\mu - \frac{\mu^2}{2}} \mathbb{E}[e^{(t-\mu)X}] = e^{t\mu - \frac{\mu^2}{2}} e^{\frac{(t-\mu)^2}{2}} = e^{\frac{t^2}{2}},$$

which is the MGF of a standard normal variable. Therefore, $Y \sim \mathcal{N}(0,1)$ in $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$.

□

We now turn our attention to the finite time horizon $[0, T]$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_T = \mathcal{F}$, and Z be a positive and $\mathcal{F}_T$-measurable random variable with $\mathbb{E}[Z] = 1$. Then, $\widetilde{\mathbb{P}}(A) := \mathbb{E}[Z\mathbb{1}_A]$, $A \in \mathcal{F}$ defines a probability measure. By Theorem 3.1, we know that for any $\mathcal{F}_T = \mathcal{F}$-measurable random variable Y, $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]$. More generally, if Y is $\mathcal{F}_t$-measurable for some $0 \leq t \leq T$, the following formula holds:

Lemma 3.2 Let Y be an $\mathcal{F}_t$-measurable random variable for some $0 \le t \le T$. Then,

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ_t],$$

where for $0 \le t \le T$,

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t].$$

Remark 3.3. By the tower property of conditional expectations, $\{Z_t\}_{t\in[0,T]}$ is a martingale; see Example 3.3 of Chapter 4.

Proof. By Theorem 3.1, we have

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}_t]] = \mathbb{E}[YZ_t],$$

where the second equality follows from the fact that Y is $\mathcal{F}_t$-measurable. □

Using the above result, we can prove the following Bayes formula for conditional expectations, which will eventually lead to Girsanov's theorem:

Lemma 3.4 (Bayes theorem for conditional expectations) Let Y be an $\mathcal{F}_t$-measurable random variable for some $t \in [0, T]$. For any $0 \le s \le t$, we have

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]. \tag{10}$$

Proof. It is clear that the RHS of (10) is $\mathcal{F}_s$-measurable. Hence, it suffices to check that, for any $A \in \mathcal{F}_s$,

$$\widetilde{\mathbb{E}}\left[\frac{\mathbb{1}_A}{Z_t}\mathbb{E}[YZ_t|\mathcal{F}_s]\right] = \widetilde{\mathbb{E}}[Y\mathbb{1}_A].$$

To show this, using Lemma 3.2 and the fact that $\frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\frac{\mathbb{1}_A}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]\right] &= \mathbb{E}\left[\left(\frac{\mathbb{1}_A}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]\right)Z_s\right] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{E}[YZ_t|\mathcal{F}_s]\right] \\
&= \mathbb{E}\left[\mathbb{1}_A YZ_t\right] \\
&= \widetilde{\mathbb{E}}[Y\mathbb{1}_A],
\end{aligned}$$

where the second-to-last equality follows from the definition of conditional expectations given $\mathcal{F}_s$, and the last equality is a consequence of Lemma 3.2 as being applied to $Y\mathbb{1}_A$. □

3.2 Girsanov's Theorem for Brownian Motions

Theorem 3.5 (Girsanov) Let $\{B_t\}_{t\in[0,T]}$ be a standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and $\{\theta_t\}_{t\in[0,T]}$ be an adapted process. Define the processes

$$\begin{aligned} Z_t &:= \exp\left(-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right) \\ \widetilde{B}_t &:= B_t + \int_0^t \theta_s \, ds. \end{aligned}$$

Suppose that

$$\mathbb{E}\left[\int_0^T \theta_t^2 Z_t^2 \, dt\right] < \infty.$$

Then, $\mathbb{E}[Z_T] = 1$, and under the probability measure $\widetilde{\mathbb{P}}$, where

$$\widetilde{\mathbb{P}}(A) := \mathbb{E}[Z_T \mathbb{1}_A], \; A \in \mathcal{F}_T = \mathcal{F},$$

the process $\{\widetilde{B}_t\}_{t\in[0,T]}$ is a standard Brownian motion.

Proof. By Lévy's characterization theorem (Theorem 3.4 in Chapter 5), it suffices to verify that $\{\widetilde{B}_t\}_{t\in[0,T]}$ is a martingale with continuous sample paths and quadratic variation $\langle \widetilde{B} \rangle_t = t$.

We first derive the quadratic variation of $\widetilde{B}$. Note that $\widetilde{B}$ is an Itô process, and by Proposition 2.3 of Chapter 5[1], we have $\langle \widetilde{B} \rangle_t = \langle B \rangle_t = t$. The continuity of $\widetilde{B}$ is clear, since B has continuous sample paths and $t \mapsto \int_0^t \theta_s \, ds$ is also continuous. Therefore, it suffices to show that $\widetilde{B}$ is a martingale under $\widetilde{\mathbb{P}}$.

To proceed, we show that

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[\widetilde{B}_t | \mathcal{F}_s] = \widetilde{B}_s$$

for any $0 \le s \le t \le T$. Using Lemma 3.4, we have

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[\widetilde{B}_t | \mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Z_t \widetilde{B}_t | \mathcal{F}_s].$$

Under measure $\mathbb{P}$, by applying Itô's lemma to the process $Z_t \widetilde{B}_t$, we have

$$\begin{aligned} d(Z_t \widetilde{B}_t) &= Z_t \, d\widetilde{B}_t + \widetilde{B}_t \, dZ_t + d\langle Z, \widetilde{B} \rangle_t \\ &= Z_t(\theta_t \, dt + dB_t) + \widetilde{B}_t(-\theta_t Z_t \, dB_t) - \theta_t Z_t \, dt \\ &= Z_t(1 - \theta_t \widetilde{B}_t) \, dB_t. \end{aligned}$$

[1] The property $\langle B \rangle_t = t$ is pathwise and therefore remains valid under $\widetilde{\mathbb{P}}$, since $\widetilde{\mathbb{P}} \ll \mathbb{P}$.

This shows that $Z_t\widetilde{B}_t$ is a martingale under $\mathbb{P}$, and thus $\mathbb{E}[Z_t\widetilde{B}_t|\mathcal{F}_s] = Z_s\widetilde{B}_s$. Consequently,

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[\widetilde{B}_t|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Z_t\widetilde{B}_t|\mathcal{F}_s] = \frac{1}{Z_s}Z_s\widetilde{B}_s = \widetilde{B}_s.$$

Therefore, $\widetilde{B}$ is a true martingale under $\widetilde{\mathbb{P}}$ and the proof is complete.

□

We now return to the discussion of the discounted stock price process, $\{D_tS_t\}_{t\in[0,T]}$, and recall that its dynamics is given by (8):

$$d(D_tS_t) = D_tS_t\sigma_t\,(\theta_t\,dt + dB_t) = D_tS_t\sigma_t\,d\widetilde{B}_t,$$

where $\widetilde{B}_t = B_t + \int_0^t \theta_s\,ds$ and $\theta_t = (\mu_t - r_t)/\sigma_t$ is the market price of risk. By Girsanov's theorem, we can define the measure $\widetilde{\mathbb{P}}$ as in Theorem 3.5 such that $\widetilde{B}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$. This measure $\widetilde{\mathbb{P}}$ is called the ***risk-neutral measure***. Some implications of the stock price process under the risk-neutral measure:

1. The discounted stock price process $\{D_tS_t\}_{t\in[0,T]}$ is a martingale under $\widetilde{\mathbb{P}}$, since the Itô's diffusion (8) is *driftless*.

2. The undiscounted stock price process $\{S_t\}_{t\in[0,T]}$ has a rate of return equal to the risk-free interest rate r_t under $\widetilde{\mathbb{P}}$:

$$\begin{aligned} dS_t &= S_t\mu_t\,dt + S_t\sigma_t\,dB_t \\ &= S_t\mu_t\,dt + S_t\sigma_t\,d\left(\widetilde{B}_t - \int_0^t \theta_s\,ds\right) \\ &= S_t(\mu_t - \sigma_t\theta_t)\,dt + S_t\sigma_t\,d\widetilde{B}_t \\ &= S_tr_t\,dt + S_t\sigma_t\,d\widetilde{B}_t. \end{aligned} \tag{11}$$

   In addition, we also have the following observations:

   (i) The volatility process σ_t of S_t is preserved under $\widetilde{\mathbb{P}}$. Since σ_t governs the dispersion and variability of the stock price paths, the set of possible realized paths of S_t remains unchanged under the risk-neutral measure.

   (ii) Since it is often the case that $\mu_t > r_t$, $\widetilde{\mathbb{P}}$ adjusts the probability weights by assigning relatively more mass to lower returns, thereby lowering the expected rate of return from μ_t to r_t.

4 Risk-Neutral Pricing

4.1 Risk-Neutral Pricing Formula

The objective is to construct a self-financing hedging portfolio $\{X_t\}_{t\in[0,T]}$ using a portfolio strategy $\{\Delta_t\}_{t\in[0,T]}$ such that

$$X_T = V_T \ \mathbb{P}\text{-a.s.}$$

By the law of one price, the initial capital X_0 would be the ***risk-neutral price*** of the security at time $t = 0$. In the next section, we shall show that such a portfolio strategy $\{\Delta_t\}_{t\in[0,T]}$ exists.

We now derive an expression for the initial capital X_0 required. Following the derivation of (1), we have

$$\begin{aligned} dX_t &= \underbrace{\Delta_t \, dS_t}_{\text{proceeds from holding the stock}} + \underbrace{r_t(X_t - \Delta_t S_t)\, dt}_{\text{proceeds from lending/borrowing at } r_t} \\ &= \Delta_t \left(S_t \mu_t \, dt + S_t \sigma_t \, dB_t\right) + r_t(X_t - \Delta_t S_t)\, dt \\ &= \Delta_t \left(S_t r_t \, dt + S_t \sigma_t \, d\widetilde{B}_t\right) + r_t(X_t - \Delta_t S_t)\, dt \\ &= r_t X_t \, dt + \Delta_t S_t \sigma_t \, d\widetilde{B}_t, \end{aligned}$$

where $\{\widetilde{B}_t\}_{t\in[0,T]}$ is the standard Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$.

Using this and the product rule, the discounted portfolio value process, $\{D_t S_t\}_{t\in[0,T]}$, where $D_t = e^{-\int_0^t r_s \, ds}$,

$$d(D_t X_t) = \Delta_t S_t \sigma_t \, d\widetilde{B}_t, \tag{12}$$

making $\{D_t S_t\}_{t\in[0,T]}$ a martingale under $\widetilde{\mathbb{P}}$. By the martingale property and the fact that $X_T = V_T$, we have

$$X_0 = D_0 X_0 = \widetilde{\mathbb{E}}\left[D_T X_T | \mathcal{F}_0\right] = \widetilde{\mathbb{E}}[D_T V_T] = \widetilde{\mathbb{E}}\left[e^{-\int_0^T r_s \, ds} V_T\right]. \tag{13}$$

The RHS (13) presents the ***risk-neutral price*** of the security at $t = 0$, which is the expected discounted value of V_T under the risk-neutral measure $\widetilde{\mathbb{P}}$.

For general $t \in [0, T]$, the value of the hedging portfolio X_t represents the ***risk-neutral price*** of the security at time t. Let $V_t = X_t$. By the martingale property of $\{D_t X_t\}_{t\in[0,T]} = \{D_t V_t\}_{t\in[0,T]}$ under $\widetilde{\mathbb{P}}$,

$$D_t V_t = \widetilde{\mathbb{E}}[D_T V_T | \mathcal{F}_t].$$

Rearranging the above yields the ***risk-neutral pricing formula***

$$V_t = \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_T | \mathcal{F}_t\right] = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r_s \, ds} V_T | \mathcal{F}_t\right].$$

4.2 Black-Scholes Formula for European Options

We derive the risk-neutral price of an European call option, $V_T = (S_T - K)_+$, where $K > 0$ denotes the strike price, under a constant risk-free rate $r_\cdot \equiv r$ and the Black-Scholes model:

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t),$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

Theorem 4.1 Under the Black-Scholes model with constant risk-free rate r, the risk-neutral price of an European call option is given by

$$V_t = c(t, S_t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}(S_T - K)_+ | \mathcal{F}_t],$$

where

$$c(t, x) = xN(d_1(T - t, x)) - Ke^{-r(T-t)}N(d_2(T - t, x)), \tag{14}$$

$N(\cdot)$ denotes the cumulative distribution function of the standard normal variable, and

$$d_1(\tau, x) := \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},$$
$$d_2(\tau, x) := \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_1(\tau, x) - \sigma\sqrt{\tau}.$$

Proof. Recall from (11) that, under the risk-neutral measure $\widetilde{\mathbb{P}}$, the stock price process follows the following SDE:

$$dS_t = S_t(r\, dt + \sigma\, d\widetilde{B}_t).$$

Solving this SDE yields

$$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\widetilde{B}_T\right),$$

which also implies

$$S_T = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(\widetilde{B}_T - \widetilde{B}_t)\right).$$

Given $\mathcal{F}_t$, we know that $Z := \frac{\widetilde{B}_T - \widetilde{B}_t}{\sqrt{T-t}} \sim N(0, 1)$ under the risk-neutral measure $\widetilde{\mathbb{P}}$. In addition,

$$S_T > K \iff Z = \frac{\widetilde{B}_T - \widetilde{B}_t}{\sqrt{T - t}} \geq \frac{\ln\left(\frac{K}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} = -d_2(T - t, S_t).$$

Therefore,

$$
\begin{aligned}
c(t,S_t) &= \widetilde{\mathbb{E}}[e^{-r(T-t)}(S_T-K)_+|\mathcal{F}_t] \\
&= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left\{S_t\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\,Z\right)-K\right\}\mathbb{1}_{\{Z>-d_2(T-t,S_t)\}}\Big|\mathcal{F}_t\right] \\
&= \int_{-d_2(T-t,S_t)}^{\infty} e^{-r(T-t)}\left\{S_t\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\,z\right)-K\right\}\varphi(z)\,dz \\
&= S_t e^{-\frac{\sigma^2(T-t)}{2}}\int_{-d_2(T-t,S_t)}^{\infty} e^{\sigma\sqrt{T-t}z}\varphi(z)\,dz - Ke^{-r(T-t)}\int_{-d_2(T-t,S_t)}^{\infty}\varphi(z)\,dz, \qquad (15)
\end{aligned}
$$

where $\varphi(z)$ is the pdf of the standard normal variable.

The first term on the right-hand side of (15) can be computed by

$$
\begin{aligned}
S_t e^{-\frac{\sigma^2(T-t)}{2}}\int_{-d_2(T-t,S_t)}^{\infty} e^{\sigma\sqrt{T-t}\,z}\varphi(z)\,dz &= S_t\int_{-d_2(T-t,S_t)}^{\infty}\frac{e^{-\frac{z^2}{2}+\sigma\sqrt{T-t}\,z-\frac{\sigma^2(T-t)}{2}}}{\sqrt{2\pi}}\,dz \\
&= S_t\int_{-d_2(T-t,S_t)}^{\infty}\frac{e^{-\frac{(z-\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}\,dz \\
&= S_t\int_{-\left(d_2(T-t,S_t)+\sigma\sqrt{T-t}\right)}^{\infty}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy \\
&= S_t\int_{-d_1(T-t,S_t)}^{\infty}\varphi(y)\,dy \\
&= S_t\left(1-N\left(-d_1(T-t,S_t)\right)\right) \\
&= S_tN(d_1(T-t,S_t)),
\end{aligned}
$$

where we have used the fact that $N(z)=1-N(-z)$ for any $z\in\mathbb{R}$. On the other hand, the second term on the right-hand side of (15) is given by

$$
\begin{aligned}
Ke^{-r(T-t)}\int_{-d_2(T-t,S_t)}^{\infty}\varphi(z)\,dz &= Ke^{-r(T-t)}\left(1-N(-d_2(T-t,S_t))\right) \\
&= Ke^{-r(T-t)}N(d_2(T-t,S_t)).
\end{aligned}
$$

Substituting these into (15), we obtain

$$
c(t,S_t) = S_tN(d_1(T-t,S_t)) - Ke^{-r(T-t)}N(d_2(T-t,S_t))
$$

as desired. □

4.3 Martingale Representation Theorem

The risk-neutral pricing formula is based on the assumption that we can find a strategy $\{\Delta_t\}_{t\in[0,T]}$ such that the portfolio value X_T agrees with the payoff of the contingent claim V_T.

The existence of such a portfolio strategy is a consequence of the martingale representation theorem:

Theorem 4.2 (Martingale Representation) Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by $\{B_t\}_{t\in[0,T]}$. Then, for any martingale $\{M_t\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$, there exists an adapted process Γ_t such that

$$M_t = M_0 + \int_0^t \Gamma_s \, dB_s.$$

Based on Theorem 4.2, we have the following martingale representation theorem upon a change of measure:

Corollary 4.3 Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by $\{B_t\}_{t\in[0,T]}$. Let $\{Z_t\}_{t\in[0,T]}$ and $\{\widetilde{B}_t\}_{t\in[0,T]}$ be the derivative process and the Brownian motion under $\widetilde{\mathbb{P}}$ as defined in Theorem 3.5, respectively. Then, for any $\{\mathcal{F}_t\}_{t\in[0,T]}$, $\widetilde{\mathbb{P}}$-martingale $\{\widetilde{M}_t\}_{t\in[0,T]}$, there exists an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process $\{\widetilde{\Gamma}_t\}_{t\in[0,T]}$ such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\Gamma}_s \, d\widetilde{B}_s.$$

Proof. We first claim that $M_t := Z_t\widetilde{M}_t$ is a $\mathbb{P}$-martingale. For any $0 \le s \le t \le T$, using Lemma 3.4,

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[Z_t\widetilde{M}_t|\mathcal{F}_s] = Z_s\mathbb{E}^{\widetilde{\mathbb{P}}}[\widetilde{M}_t|\mathcal{F}_s] = Z_s\widetilde{M}_s,$$

which verifies the claim. By Theorem 4.2, there exists an adapted process Γ such that

$$M_t = M_0 + \int_0^t \Gamma_s \, dB_s.$$

To proceed, we apply Itô's lemma on $\widetilde{M}_t := \frac{M_t}{Z_t}$. Recall

$$dZ_t = -Z_t\theta_t \, dB_t,$$

so that

$$\begin{aligned} d\left(\frac{1}{Z_t}\right) &= -\frac{dZ_t}{Z_t^2} + \frac{1}{2}\left(\frac{2}{Z_t^3}\right) d\langle Z\rangle_t \\ &= \frac{\theta_t Z_t}{Z_t^2} \, dB_t + \frac{Z_t^2\theta_t^2}{Z_t^3} \, dt \end{aligned}$$

$$= \frac{1}{Z_t}\left(\theta_t^2\, dt + \theta_t\, dB_t\right).$$

Hence, by the product rule,

$$\begin{aligned}
d\widetilde{M}_t &= d\left(\frac{M_t}{Z_t}\right) \\
&= \frac{dM_t}{Z_t} + M_t d\left(\frac{1}{Z_t}\right) + d\left\langle M, \frac{1}{Z_t}\right\rangle \\
&= \frac{\Gamma_t}{Z_t}\, dB_t + \frac{M_t}{Z_t}\left(\theta_t^2\, dt + \theta_t\, dB_t\right) + \frac{\theta_t \Gamma_t}{Z_t}\, dt \\
&= \frac{\theta_t}{Z_t}\left(M_t\theta_t + \Gamma_t\right)\, dt + \frac{1}{Z_t}\left(\Gamma_t + \theta_t M_t\right)\, dB_t \\
&= \frac{\theta_t}{Z_t}\left(M_t\theta_t + \Gamma_t\right)\, dt + \frac{1}{Z_t}\left(\Gamma_t + \theta_t M_t\right)\, \left[d\widetilde{B}_t - \theta_t\, dt\right] \\
&= \frac{\Gamma_t + \theta_t M_t}{Z_t}\, d\widetilde{B}_t \\
&= \left(\frac{\Gamma_t}{Z_t} + \widetilde{M}_t\right)\, d\widetilde{B}_t.
\end{aligned}$$

Therefore, we arrive at the representation with

$$\widetilde{\Gamma}_t := \frac{\Gamma_t}{Z_t} + \widetilde{M}_t.$$

□

The existence of $\{\Delta_t\}_{t\in[0,T]}$ can now be proven using Corollary 4.3. From (12), we know that $\{D_t X_t\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale. By Corollary 4.3, there exists an adapted process $\widetilde{\Gamma}$ such that

$$D_t X_t = X_0 + \int_0^t \widetilde{\Gamma}_s\, d\widetilde{B}_s.$$

On the other hand, by (12), we also have

$$D_t X_t = X_0 + \int_0^t \Delta_s S_s \sigma_s\, d\widetilde{B}_s.$$

By equating the coefficient of the diffusion term, we arrive at

$$\Delta_t = \frac{\widetilde{\Gamma}_t}{S_t \sigma_t}.$$